https://www.modscires.pro/index.php/msr/article/view/be3-118-049
DOI: 10.30889/2523-4692.2018-03-01-049

УДК 51-7(075.8)

THE PROBABILITY OF BANKRUPTCY IN THE CASE OF “HEAVY TAILS” AND THE ADMISSIBLE INCSURANCE RATE.

ЙМОВІРНІСТЬ БАНКРУТСТВА У ВИПАДКУ «ВАЖКИХ ХВОСТІВ» І ОПТИМАЛЬНА СТРАХОВА СТВАВКА

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Abstract. *In this paper we consider the problem of defining the asymptotic of the bankruptcy probability in the case of large payments distributed under subexponential laws and defining an admissible insurance rate for F-model case.*

Key words: *Bankruptcy probability asymptotics, heavy tails, subexponential distributions, F-Model, admissible incsurance rate.*

Introduction

Classical risk theory assumes that large insurance claims and, therefore, large insurance payments are rare, with exponentially small probabilities. This scheme is called "model with small payments".

However, many situations are related to extreme events. Due to this fact, the true size of payments is more adequately represented by random variables distributed with "heavy tails", which include Pareto type distributions. Actually, in this case, total payments will be defined by the maximal individual claim. This effect became extremely noticeable at the beginning of the 2000s, when the insurance companies had to refund significant amounts on insurance claims, conditioned by catastrophes: earthquakes, fires, floods, terrorist attacks.

Suppose that we are in the classic problem of finding the probability of bankruptcy ([1], 184-186, [2], [3]).. Let

1) $\varphi(u,T) = P\ \{U(t)<0$ for some $0<t\leq T\}, 0<t<\infty$, $u>0$– probability of bankruptcy on a finite time interval $[0,T]$, $U(t)$ - the process of risk;

2) $\varphi(u)=\varphi(u,\infty)= P\ \{U(t)<0$ for some $t>0\}$ – probability of bankruptcy on an infinite interval.

To calculate the probability of bankruptcy it is comfortable to have a simple analytical formulas for $\varphi(u)$ or $\varphi(u,T)$ that include probabilistic characteristics of the insurance payments and process flow requirements for payment $N(t)$. First, we need the distribution function if $F(x)$.

As a next step we will review the asymptotical behavior $\varphi(u)$ when the initial capital u arises and the distribution function $F(x)$ amount of large payments satisfies some additional conditions.

However, it worth mentioning that the bankruptcy occurs only when the payment claims T_n arise.

Will use the following terms and symbols: if $F(x)$ - distribution function, $\overline{F}(x) = 1 - F(x)$ the "tail" of the distribution F, and a F^{n^*} - n-fold convolution F.

So if F - distribution function of benefits, then $\overline{F}(x)$ - "tail" of the distribution, and

$$F_I(x) = \frac{1}{\mu}\int_0^x \overline{F}(y)dy,\ x > 0,$$

called integrated "tail" of the distribution. [4 p.186]

The value of $\rho = \frac{c}{\lambda\mu} - 1$ called the relative insurance premium and for the basic conditions:

$$\rho = \frac{c}{\lambda\mu} - 1 > 0 \tag{1}$$

we use the term "a net profit". In condition (1) $\varphi(u)$ can be written as ([1], p. 187):

$$\varphi(u) = \frac{\rho}{1+\rho}\sum_{n=0}^{\infty}(1+\rho)^{-n}(1 - F_I^{n_*}(u)),$$

Additional conditions for F provide us exponential estimations for $\varphi(u)$. Cramer-Lundberg's condition implies the existence of constants ν called debug (regulator, adjusting) ratio or Lundberg's ratio, such that

$$\int_0^{\infty} e^{\nu x}(1 - F(x))dx = c/\lambda = (1+\rho)\mu, \tag{2}$$

The distributions that do not satisfy the condition (2) will be called distributions with "heavy tails" [1 p.188].

Bankruptcy probability for subexponential distributions.

Let's assume that the distribution function $F(x), x \in R_+ = [0,\infty)$ satisfies the condition $F(x) < 1\ \forall x \in R_+$.

Definition. Let's call the distribution function F subexponential if for all $n \geq 2$

$$\lim_{x\to\infty}\frac{\overline{F^{n^*}}(x)}{\overline{F}(x)} = \lim_{x\to\infty}\frac{1 - F^{n^*}(x)}{1 - F(x)} = n.$$

Class of subexponential functions will be marked as S. [1, p. 189].

Note that subexponential distributions were introduced by Chistyakov [4] in context of theory of branching processes.

For further practical implementation of calculation of bankruptcy probability we use the following theorem (see in particular [1, p. 197]).

Theorem. Consider Cramer-Lundberg model under the conditions $\rho > 0$ and $F_I(\mathrm{x}) \in S$. Then

$$\varphi(u) \sim \rho^{-1}\overline{F_I}(u), u \to \infty \tag{3}$$

According to this theorem, in the case of payments which have distributions of subexponential integrated "tails", the probability of bankruptcy allows a simple approximation, given by the formula (3).

Note that the condition of the theorem formulated in terms integrated "tails" instead of the distribution function F(x). Logical question arises: if $F_I(x) \in S$ follows from $F(x) \in S$, or vice versa? General answer is – NO. Thus, there are defined distributions for which we can calculate the bankruptcy probability.

Calculating the probability of bankruptcy for large payments

Consider the problem of calculating the probability of bankruptcy in the case of "heavy tails", that is, when payments are large. Note that in the case of Pareto distribution with distribution function $F(x) = 1 - \left(\frac{k}{x}\right)^{\alpha}, \alpha > 1, k > 0, x > 0$ and include the log-normal distribution considered in [1, p.198-199].

Statements 1. When payments have Pareto distribution, that is :

$$F(x) = 1 - \left(\frac{k}{k+x}\right)^{\alpha}, \alpha > 1, k > 0, x > 0.$$

Then the asymptotic of bankruptcy probability $\phi(u)$ is defined as:

$$\phi(u) \sim \frac{\lambda \mathrm{k}^{\alpha}}{c(\alpha-1)-\lambda k}(k+u)^{-\alpha+1}, u \to \infty.$$

Proof: The distribution density is:

$$f(x) = \frac{\alpha k}{(k+x)^{\alpha+1}}.$$

In this case, the mathematical value: $\mu = EX_1 = \frac{k}{(\alpha-1)}$.

Then relative insurance premium:

$$\rho = \frac{c}{\lambda\mu} - 1 = \frac{c(\alpha-1)}{\lambda k} - 1; \rho^{-1} = \frac{\lambda k}{c(\alpha-1)-\lambda k}.$$

If $F(x)$ – a distribution function of payment amount, then $\overline{F}(x)$ – «tail» of this distribution.

$$\overline{F}(x) = \left(\frac{k}{k+x}\right)^{\alpha}, x > 0.$$

Since, the integrated “tail” of distribution:

$$F_I(x) = \frac{1}{\mu}\int_0^x \overline{F}(y)dy, x > 0.$$

Then

$$\int_0^x \overline{F}(y)dy = \int_0^x \left(\frac{k}{k+y}\right)^{\alpha} dy = k^{\alpha} \frac{(k+y)^{-\alpha+1}}{-\alpha+1}\Bigg|_0^x = k^{\alpha}\left(\frac{(k+x)^{-\alpha+1}}{-\alpha+1} - \frac{k^{-\alpha+1}}{-\alpha+1}\right) =$$

$$\frac{k}{\alpha-1}\left(1-k^{\alpha-1}(k+x)^{-\alpha+1}\right).$$

$$F_I(x) = 1-k^{\alpha-1}(k+x)^{-\alpha+1}; 1-F_I(x) = k^{\alpha-1}(k+x)^{-\alpha+1}.$$

Therefore, bankruptcy probability asymptotic is defined as:

$$\phi(u) \sim \frac{\lambda \mathrm{k}^{\alpha}}{c(\alpha-1)-\lambda k}(k+u)^{-\alpha+1},\ u \to \infty.$$

According Pareto distribution analysis it is obvious that this is a distribution whose tail is heading to zero as x^a that leads to distribution with tail that is much heavier than the exponential one. Let's consider the right tails of this distributions:

1. Exponential $P(X > x) = \exp(-\lambda x)$
2. Pareto $P(X > x) = (\lambda / (\lambda + x)^a)$

Consider:

$$P(X > x) = \exp(-cx^{\gamma}), \gamma > 0.$$

Now we have two cases. If $\gamma < 1$, then arises one more distribution that is between Pareto and exponential distributions. At the same time in case $\gamma > 1$ the right tale is lighter then the exponential one($\gamma = 1$corresponds to exponential distribution). This behavior of the tails definers the Weibull distribution as very flexible and the one that can be used in insurance problems for modeling losses(usually with $\gamma < 1$). Let's consider the next statement for the Weibull distribution.

Statements 2. Let payments distributed by Weibull distribution with a parameter $0 < \gamma < 1$, and the distribution function

$$F(x) = 1 - \exp\left(-c_1 x^{\gamma}\right),\ c_1 > 0, x > 0$$

then the asymptotic of probability of bankruptcy is given

$$\varphi(u) \sim \frac{\lambda}{\mathrm{c} \cdot \mathrm{c}_1^{\frac{1}{\gamma}} - \lambda \Gamma\left(1+\frac{1}{\lambda}\right)}\left[1+\frac{\Gamma\left(\frac{1}{\gamma}; c_1 x^{\gamma}\right) - \Gamma\left(\frac{1}{\gamma}; 0\right)}{\gamma \cdot \Gamma\left(1+\frac{1}{\gamma}\right)}\right],\ u \to \infty. \quad (4)$$

Proof: Weibull distribution density:

$$f(x) = c_1 \gamma x^{\gamma-1} \exp\left(-c_1 x^{\gamma}\right).$$

The mathematical expectation: $\mu = EX_1 = \frac{1}{c^{\frac{1}{\gamma}}} \cdot \Gamma\left(1+\frac{1}{\gamma}\right).$

Then relative insurance premium:

$$\rho=\frac{c_1}{\lambda\mu}-1=\frac{c_1\cdot c^{\frac{1}{\gamma}}}{\lambda\Gamma\left(1+\frac{1}{\gamma}\right)}-1=\frac{c_1\cdot c^{\frac{1}{\gamma}}-\lambda\Gamma\left(1+\frac{1}{\gamma}\right)}{\lambda\Gamma\left(1+\frac{1}{\gamma}\right)}.$$

Then $\rho^{-1}=\dfrac{\lambda\Gamma\left(1+\frac{1}{\gamma}\right)}{c\cdot c_1^{\frac{1}{\gamma}}-\lambda\Gamma\left(1+\frac{1}{\gamma}\right)}.$

The integrated distribution tail :

$$F_I\left(x\right)=\frac{1}{\mu}\int_0^x\overline{F}(y)dy,\ x>0.$$

Note that the integration boundaries are fixed in case of classical integral definition of gamma function. Also it is considered an incomplete gamma function that is defined by the same integral with variable upper and lower boundary. There is also defined upper incomplete gamma function:

$$\Gamma\left(a,z\right)=\int_z^\infty e^{-t}t^{a-1}dt$$

Then

$$\int_0^x\overline{F}(y)dy=\int_0^x\exp\left(-c_1y^\gamma\right)dy=\frac{c_1^{-\frac{1}{\gamma}}\left(\Gamma\left(\frac{1}{\gamma};c_1x^\gamma\right)-\Gamma\left(\frac{1}{\gamma};0\right)\right)}{\gamma}.$$

Remind that: $\Gamma\left(a,z\right)=\int_z^\infty\exp(-t)t^{a-1}dt.$

As s result it is obvious that for Weibull distribution the statement (4) is executed.

Statement 3. Let payments distributed Benktander type I:

$$1-F(x)=\left(1+\frac{2\beta\ln x}{\alpha}\right)x^{-(\alpha+1+\beta\ln x)}\ \alpha,\beta>0,x>1,$$

Asymptotic of probability of bankruptcy $\varphi(u)$ given by the following equation:

$$\varphi(u)\sim\frac{\lambda(\alpha+1-\mathrm{u}^{-\alpha-\beta\ln x})}{c\alpha-\lambda(\alpha+1)},u\to\infty.$$

Proof: The function of distribution and density are as follows:

$$F(x)=1-\left(1+\frac{2\beta\ln x}{\alpha}\right)x^{-(\alpha+1+\beta\ln x)}\ \alpha,\beta>0,x>1,$$

$$f(x)=\left(\left[\left(1+\frac{2\beta\ln x}{\alpha}\right)(1+\alpha+2\beta\ln x)\right]-\frac{2\beta}{\alpha}\right)x^{-(2+\alpha+\beta\ln x)}.$$

Find the mathematical expectation: $\mu = EX = \int_1^{+\infty} xf(x)dx = \dfrac{\alpha+1}{\alpha}$

We find $\rho^{-1} = \dfrac{\lambda(\alpha+1)}{c\alpha - \lambda(\alpha+1)}$.

Find the integrated "tail" of the distribution:

$$F_I(x) = \frac{1}{\mu}\int_0^x \overline{F}(y)dy = \frac{x^{-\alpha-\beta\ln x}}{\alpha+1}$$

Accordingly, the asymptotic of probability of bankruptcy is given by the following equation:

$$\varphi(u) \sim \rho^{-1}\overline{F_I}(u) \sim \frac{\lambda(\alpha+1\text{ - u}^{-\alpha-\beta\ln x})}{c\alpha - \lambda(\alpha+1)}, u \to \infty.$$

Statement 4. . Let payments distributed Benktander type II [1, p. 196]:

$$1 - F(x) = \exp\left(\frac{\alpha}{\beta}\right)x^{-(1-\beta)}\exp\left\{-\frac{\alpha x^\beta}{\beta}\right\}, \alpha, \beta > 0, x > 1,$$

Asymptotic of probability of bankruptcy $\varphi(u)$ given by the following equation:

$$\phi(u) \sim \rho^{-1}\overline{F_I}(u) \sim \frac{\lambda}{\left(\alpha\text{c-}\lambda(1+\alpha)\right)}\exp\left(\frac{\alpha}{\beta}\right)\exp\left(-\frac{\alpha u^\beta}{\beta}\right), u \to \infty.$$

Proof: The function of distribution and density are as follows:

$$F(x) = 1 - \exp\left(\frac{\alpha}{\beta}\right)x^{-(1-\beta)}\exp\left\{\frac{-\alpha x^\beta}{\beta}\right\}, \alpha, \beta > 0, x > 1;$$

$$f(x) = \exp\left(\frac{\alpha}{\beta}\right)(1-\beta)\cdot x^{\beta-2}\exp\left\{\frac{-\alpha x^\beta}{\beta}\right\} + \exp\left(\frac{\alpha}{\beta}\right)\alpha x^{2\beta-2}\exp\left\{\frac{-\alpha x^\beta}{\beta}\right\}.$$

Mathematical expectation: $\mu = EX_1 = \dfrac{1+\alpha}{\alpha}$.

Then: $\rho = \dfrac{c}{\lambda\mu} - 1 = \dfrac{c\alpha}{\lambda(1+\alpha)} - 1 > 0$.

As a result: $\rho^{-1} = \dfrac{\lambda\mu}{(c-\lambda\mu)} = \dfrac{\lambda(1+\alpha)}{\left(\alpha c - \lambda(1+\alpha)\right)}$.

Thus, if $F(x)$ – distribution function of payment amount, then $\overline{F}(x)$ – tail of this distribution

$$\overline{F}(x) = \exp\left(\frac{\alpha}{\beta}\right)x^{-(1-\beta)}\exp\left\{-\frac{\alpha x^\varphi}{\beta}\right\} = \begin{cases} \exp\left(\frac{\alpha}{\beta}\right)x^{-(1-\beta)}\exp\left\{-\frac{\alpha x^\beta}{\beta}\right\}, x > 1, \\ 1, x \le 1. \end{cases}$$

Find the integrated "tail" of the distribution:

$$F_I(x) = \frac{1}{\mu}\int_0^x \overline{F}(y)dy, \; x > 0.$$

Hence,

$$\int_0^x \overline{F}(y)dy = \int_0^1 dy + \int_1^x \exp\left(\frac{\alpha}{\beta}\right) y^{-(1-\beta)} \exp\left\{-\frac{\alpha y^{\beta}}{\beta}\right\} dy = 1 + \frac{1}{\alpha} - \frac{\exp\left(\frac{\alpha}{\beta}\right)}{\alpha} \exp\left\{-\frac{\alpha x^{\beta}}{\beta}\right\},$$

$x > 1$.

As a result we get:

$$F_I(x) = 1 - \frac{\exp\left(\frac{\alpha}{\beta}\right)}{(1+\alpha)} \exp\left\{-\frac{\alpha x^{\beta}}{\beta}\right\}.$$

$$\overline{F_I}(x) = 1 - F_I(x) = \frac{\exp\left(\frac{\alpha}{\beta}\right)}{(1+\alpha)} \exp\left\{\frac{-\alpha x^{\beta}}{\beta}\right\}.$$

Accordingly, bankruptcy probability asymptotic is given in next way:

$$\phi(u) \sim \rho^{-1} \overline{F_I}(u) \sim \frac{\lambda}{(\alpha c - \lambda(1+\alpha))} \exp\left(\frac{\alpha}{\beta}\right) \exp\left(-\frac{\alpha u^{\beta}}{\beta}\right), u \to \infty.$$

Defining asymptotic of admissible insurance rate in case of F-model

Important moment of insurance mathematics is a definition of insurance fee that provides us with non-bankruptcy probability under some conditions. Also, interesting thing is that payments might be big enough being described by distributions with heavy tails. Asymptotical behavior of optimal insurance rate as found in case of big payments and factorization model (F-model) conditions [7].

If u_0 is a starting capital, then the final insurance fund is:

$$U = u_0 + \overline{Z} - \overline{Y} \tag{5}$$

The first problem related to (5) is definition of distribution asymptotic of random variable U in case if z is known.

Second problem – is defining a minimal value for z that will provide us with acceptable results of insurance practice for that particular insurance portfolio. These particular problems are considered in [8].

In particular it follows from [8] that in case of big payments, that have Pareto distribution with such $a > 0, \lambda > 0$ parameters that :

$$F(x) = 1 - \left(\frac{\lambda}{\lambda + x}\right)^a, x > 0,$$

z_0 - admissible insurance rate satisfies next relation:

$$z_0 \sim \frac{\lambda}{a-1} + \frac{\sqrt{\frac{a\lambda^2}{(a-1)^2(a-2)}}[1+V^2]^{1/2}\Psi(Q)}{[N - V^2\Psi^2(Q)]^{1/2}}$$

See details in [8].

Conclusion

Problem of bankruptcy probability definition in case of heavy tails has been considered. Asymptotic of bankruptcy probability in case of payments with Pareto, Weibull, Benktander type I and II distributions have been defined. As a result we received asymptotic of admissible insurance rate in case of F-model for relational claims with Pareto distribution.

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***Анотація**. Розглянуто задачу знаходження асимптотики ймовірності банкрутства у випадку великих виплат розподілених за субекспоненційними законами та визначення оптимальної страхової ставки.*

***Ключові слова:** асимптотика ймовірності банкрутства, важкі хвости, субекспоненційні розподіли, F-модель, оптимальна страхова ставка.*

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Статья отправлена: 14.03.2018 г.